

# Electron-positron pair productions in classical electric field and electromagnetic wave

Hagen Kleinert<sup>(a,b)</sup>, and She-Sheng Xue<sup>(b,c)</sup>

<sup>(a)</sup>*Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin, Germany*

<sup>(b)</sup>*ICRANeT Piazzale della Repubblica, 10 -65122, Pescara and*

<sup>(c)</sup>*Physics Department, University of Rome “La Sapienza”, P.le A. Moro 5, 00185 Rome, Italy*

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Using semiclassical WKB-methods, we calculate the pair-production rate in the presence of two external fields, a strong (space- or time-dependent) classical field and a monochromatic electromagnetic wave. We discuss Pauli-suppression and Bose-enhancement of the rate in the presence of thermal electrons and bosons at a given temperature and chemical potential. Using our rate formula, we calculate the rate enhancement due to laser and thermal photons, and discuss the possibility that the enhancement may be significant in a plasma of electrons and protons with self-focusing properties.

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## I. INTRODUCTION

The creation of electron-positron pairs from the vacuum by an external uniform electric field in spacetime was first studied by Sauter [1] as a quantum tunneling process. Heisenberg and Euler [2] extended his result by calculating an effective Lagrangian from the Dirac theory for electrons in a constant electromagnetic field. A more elegant reformulation was given by Schwinger [3] based on *Quantum Electrodynamics* (QED), where the result is obtained from a one-loop calculation of the electron field in a constant electromagnetic field yielding an effective action. A detailed review and relevant references can be found in Refs. [4] and [5].

The rate of pair-production may be split into an exponential and a pre-exponential factor. The exponent is determined by the classical trajectory of the tunneling particle in imaginary time which has the smallest action. It plays the same role as the activation energy in a Boltzmann factor with a “temperature”  $\hbar$ . The pre-exponential factor is determined by the quantum fluctuations of the path around that trajectory. At the semiclassical level, the latter is obtained from the functional determinant of the quadratic fluctuations. It can be calculated in closed form only for a few classical paths [6]. An efficient technique for doing this is based on the WKB wave functions, another on solving the Heisenberg equations of motion for the position operator in the external

field [6]. If the electric field depends only on time, both exponential and pre-exponential factors were approximately computed by Brezin and Itzykson by applying Schwinger's method to a purely periodic field  $E(t) = E_0 \cos \omega_0 t$  [7]. The result was generalized by Popov with a First-quantized calculation in Ref. [8] to a general time-dependent fields  $E(t)$ . An alternative approach to the same problems was more recently employed using the so-called worldline formalism [9], sometimes called the "string-inspired formalism". This formalism is closely related to Feynman's orbital view of the propagators of quantum fields. The functional determinant of the electron field in Schwinger's approach is calculated as a relativistic path integral over all fluctuating orbits of an electron in the external field as described in the textbook [6]. In the path integral formalism the tunneling problem has a standard formulation and the pre-exponential factor is calculated via an orbital fluctuation determinant for whose calculation simple formulas have been developed in Ref. [6]. These formula were evaluated by Dunne and Schubert [10] and Dunne et al. [11] for various field configurations, such as the single-pulse field with a temporal Sauter shape  $\propto 1/\cosh 2\omega t$ .

In our previous article [12], we have derived a general expression for the pair-production rate in nonuniform electric fields  $E(z)$  pointing in the  $z$ -direction and varying only along this direction. A simple variable change in all formulas has led to results for electric fields depending on time rather than space.

The relevant critical field where copious pair production sets in is  $E_c = m_0 c^2/e = 1.3 \times 10^{16} \text{ V/cm}$ , field intensity  $I_c = E_c^2/2 \simeq 4.3 \times 10^{29} \text{ W/cm}^2$ . For electric fields  $E \ll E_c$ , the pair-production rate is exponentially reduced by a factor  $\exp -\pi E_c/E$ . In the laboratory, the electric field intensity  $I_c$  is, unfortunately, extremely difficult to reach in the laboratory [13, 14]. Motivated by these difficulties, people have studied possibilities of a dynamical enhancement of pair-production rate by time-dependent oscillating or pulse electric fields [15, 16].

One possibility is to consider the superposition of a strong but slow field pulse and a weak but fast field pulse, which can lead to a significant enhancement of pair-production rate [15, 17]. Another is a catalysis mechanism of pair productions that has been studied in Ref. [16]. The setup is a superposition of a plane-wave X-ray probe beam with a strongly focused optical laser pulse. Namely, the optical laser pulse beams are focused onto a spot to yield a strong stationary electric field  $E$ , and the X-ray laser propagates through the focusing spot of optical laser beams. Since X-ray laser wavelength (frequency) is much smaller (larger) than the optical one, namely the size of the focusing spot, the electric field created by focusing two optical laser beams can be approximated by a constant classical electric field in space and time. In that spot, a large number of coherent photons (X-ray laser) collides with virtual pairs of the vacuum in a strong classical electric field

(optical intense pulse), and in consequence the pair-production rate must be enhanced.

In this article, we continue and extend our semiclassical WKB-approach [12], by calculating the enhanced pair-production rate in the superposition of a strong (space- or time-dependent) classical field and an electromagnetic plane wave. In Sections II and III, we present a general expression for the rate with a general enhancement factor. In Section IV we apply this general expression to two cases; (i) a constant electric field in finite spatial region which drops sharply to zero at the boundary; (ii) a softened version of this, where the production takes place in a Sauter electric step field. In Section V, we extend our general formalism by calculating the enhancement factor in the presence of coherent laser photons and thermal photons at a finite temperature. In Sections VI and VII, we discuss the Pauli-suppression and Bose-enhancement of pair-productions rate in the presence of thermal electrons and bosons at a given temperature and chemical potential. Finally, in Section VIII, discuss the possibility that the pair-production rate can be greatly enhanced by the self-focusing phenomenon of laser beam propagating in the plasma of electrons and protons.

## II. SEMICLASSICAL DESCRIPTION OF PAIR PRODUCTION

The phenomenon of pair production in an external electric field can be understood, in the historic first-quantized Dirac picture, as a quantum-mechanical tunneling process of electrons from the negative-energy Dirac sea to the positive energy conduction band [18, 19]. The electric field bends the positive and negative-energy levels of the Hamiltonian, leading to a level-crossing and a tunneling of the electrons in the negative-energy band to the positive-energy band. Let the field vector  $\mathbf{E}(z)$  point in the  $z$ -direction. In the one-dimensional potential energy

$$V(z) = -eA_0(z) = e \int^z dz' E(z') \quad (1)$$

of an electron of charge  $-e$ , the classical positive and negative-energy spectra are

$$\mathcal{E}_{\pm}(p_z, p_{\perp}; z) = \pm \sqrt{(cp_z)^2 + c^2 p_{\perp}^2 + (m_e c^2)^2} + V(z), \quad (2)$$

where  $p_z$  is the momentum in the  $z$ -direction,  $\mathbf{p}_{\perp}$  the momentum orthogonal to it, and  $p_{\perp} \equiv |\mathbf{p}_{\perp}|$ . For a given energy  $\mathcal{E}$ , the tunneling takes place from  $z_-$  to  $z_+$  determined by  $p_z = 0$  in Eq. (2)

$$\mathcal{E} = \mathcal{E}_+(0, p_{\perp}; z_+) = \mathcal{E}_-(0, p_{\perp}; z_-). \quad (3)$$

The points  $z_{\pm}$  are the *turning points* of the classical trajectories crossing from the positive-energy band to the negative one at energy  $\mathcal{E}$ . They satisfy the equations

$$V(z_{\pm}) = \mp \sqrt{c^2 p_{\perp}^2 + m_e^2 c^4} + \mathcal{E}. \quad (4)$$

This energy-level crossing  $\mathcal{E}$  is shown in Fig. 1 for the Sauter potential  $V(z) \propto \tanh(z/\ell)$ .

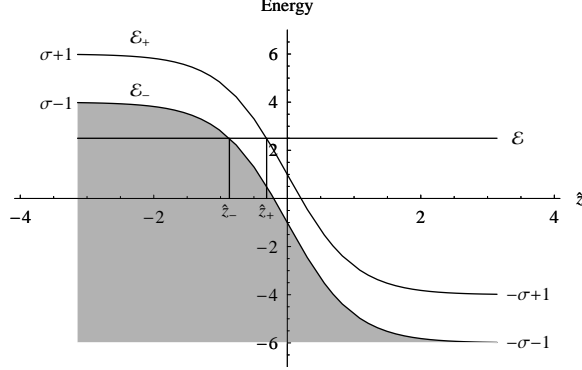


FIG. 1: Positive- and negative-energy spectra  $\mathcal{E}_{\pm}(z)$  of Eq. (2) in units of  $m_e c^2$ , with  $p_z = p_{\perp} = 0$  as a function of  $\hat{z} = z/\ell$  for the Sauter potential  $V_{\pm}(z)$  (77) for  $\sigma = 5$ . This figure is reproduced from Fig. 1 in Ref. [12].

### A. WKB transmission probability for Klein-Gordon Field

In our previous article [12], using the WKB method we obtain the general formula for the electron-positron pair production rate in space- or time-dependent electric fields. In this article, using this approach, we further study the electron-positron pair production rate in the presence of radiation (photon) and electron fields. First we consider a monochromatic radiation field, which is described as an electromagnetic plane wave  $A_{\mu}$ ,

$$A_{\mu}(x) = [A_0, \mathbf{A}_{\perp}(t, z), A_z(t)], \quad A_0 = 0, \quad A_z = 0, \quad \partial_{\mu} A^{\mu} = k_{\mu} A^{\mu} = 0 \quad (5)$$

with the wave vector  $k_{\mu} = (\omega, \mathbf{k})$ ,  $k^2 = 0$ , propagating in the  $\hat{z}$ -direction  $k_z \neq 0, k_{\perp} = 0$ ,

$$\mathbf{A}_{\perp}(t, z) = \mathbf{A}_{\perp}(k) \exp \frac{i}{\hbar} (p_z^{\gamma} z - \mathcal{E}^{\gamma} t) \quad (6)$$

where radiation photon energy-momentum  $\mathcal{E}_{\gamma} = \hbar\omega$ ,  $p_z^{\gamma} = \hbar k_z$  and  $\mathcal{E}^{\gamma} = c|p_z^{\gamma}|$ .

The probability of quantum tunneling in the  $z$ -direction is most easily studied for a scalar field which satisfies the Klein-Gordon equation

$$\left\{ \left[ i\hbar\partial_{\mu} + \frac{e}{c} A_{\mu}^{*}(z) \right]^2 - m_e^2 c^2 \right\} \phi(x) = 0, \quad (7)$$

where  $x_0 \equiv ct$ , and the gauge potential  $A_\mu^*(z)$  is superposition of two parts: the static field  $A_\mu(z) = (A_0(z), 0, 0, 0)$  of Eq. (1) and an electromagnetic plane wave  $A_\mu$  of Eq. (5). Then Eq. (7) becomes

$$\left\{ \left[ i\hbar\partial_\mu + \frac{e}{c}A_\mu(z) \right]^2 - m_e^2c^2 + 2i\hbar\frac{e}{c}(A_\mu\partial^\mu) + \frac{e^2}{c^2}A^2 \right\} \phi(x) = 0, \quad (8)$$

where  $A_\mu A^\mu = 0$  for the gauge chosen. Since there are only the static electric field in the  $\hat{z}$ -direction and electromagnetic plane-wave fields in the perpendicular  $\hat{x}, \hat{y}$ -directions, both vary only along  $z$ , we can choose a vector potential with the only nonzero component (1), and make the ansatz:

$$\phi(x) = e^{-i\mathcal{E}t/\hbar} e^{i\mathbf{p}_\perp \mathbf{x}_\perp/\hbar} \phi_{\mathbf{p}_\perp, \mathcal{E}}(z) \chi_{\mathbf{p}_\perp, \mathcal{E}}(z), \quad (9)$$

a fixed momentum  $\mathbf{p}_\perp$  in the  $x, y$  direction and an energy  $\mathcal{E}$ , and Eq. (8) becomes simply

$$\left\{ -\hbar^2 \frac{d^2}{dz^2} + p_\perp^2 + m_e^2c^2 - \frac{1}{c^2} [\mathcal{E} - V(z)]^2 + 2\frac{e}{c}(\mathbf{A}_\perp \mathbf{p}_\perp) - \frac{e^2}{c^2} A^2 \right\} \phi_{\mathbf{p}_\perp, \mathcal{E}}(z) \chi_{\mathbf{p}_\perp, \mathcal{E}}(z) = 0. \quad (10)$$

By expressing the wave function  $\phi_{\mathbf{p}_\perp, \mathcal{E}}(z)$  as an exponential

$$\phi_{\mathbf{p}_\perp, \mathcal{E}}(z) = \mathcal{C} e^{iS_{\mathbf{p}_\perp, \mathcal{E}}/\hbar}, \quad (11)$$

where  $\mathcal{C}$  is some normalization constant, we impose the wave equation for  $\phi_{\mathbf{p}_\perp, \mathcal{E}}(z)$

$$\left[ -\hbar^2 \frac{d^2}{dz^2} + p_\perp^2 + m_e^2c^2 - \frac{1}{c^2} [\mathcal{E} - V(z)]^2 \right] \phi_{\mathbf{p}_\perp, \mathcal{E}}(z) = 0, \quad (12)$$

which becomes a Riccati equation for  $S_{\mathbf{p}_\perp, \mathcal{E}}$ :

$$-i\hbar\partial_z^2 S_{\mathbf{p}_\perp, \mathcal{E}}(z) + [\partial_z S_{\mathbf{p}_\perp, \mathcal{E}}(z)]^2 - p_z^2(z) = 0. \quad (13)$$

where the function  $p_z(z)$  is the solution of the equation

$$p_z^2(z) = \frac{1}{c^2} [\mathcal{E} - V(z)]^2 - p_\perp^2 - m_e^2c^2. \quad (14)$$

The solution of Eq. (13) can be found iteratively as an expansion in powers of  $\hbar$ :

$$S_{\mathbf{p}_\perp, \mathcal{E}}(z) = S_{\mathbf{p}_\perp, \mathcal{E}}^{(0)}(z) - i\hbar S_{\mathbf{p}_\perp, \mathcal{E}}^{(1)}(z) + (-i\hbar)^2 S_{\mathbf{p}_\perp, \mathcal{E}}^{(2)}(z) + \dots \quad (15)$$

Neglecting the expansion terms after  $S_{\mathbf{p}_\perp, \mathcal{E}}^{(1)}(z) = -\log p_z^{1/2}(z)$  leads to the WKB approximation for the wave functions of positive and negative energies can (see e.g. [20, 21])

$$\phi_{\mathbf{p}_\perp, \mathcal{E}}^{\text{WKB}}(z) = \frac{\mathcal{C}}{p_z^{1/2}(z)} e^{iS_{\mathbf{p}_\perp, \mathcal{E}}^{(0)}(z)/\hbar}. \quad (16)$$

where  $S_{\mathbf{p}_\perp, \mathcal{E}}^{(0)}(z)$  is the eikonal

$$S_{\mathbf{p}_\perp, \mathcal{E}}^{(0)}(z) = \int^z p_z(z') dz'. \quad (17)$$

Following the differential equations (10) and (12), the differential equation for the function  $\chi_{\mathbf{p}_\perp, \mathcal{E}}(\varphi)$  is given by

$$\left[ -2\hbar^2 \eta^z \frac{d}{dz} + 2\frac{e}{c}(\mathbf{A}_\perp \mathbf{p}_\perp) - \frac{e^2}{c^2} \mathbf{A}_\perp^2 \right] \chi_{\mathbf{p}_\perp, \mathcal{E}}(z) = 0. \quad (18)$$

where we use  $\partial^\mu F = k^\mu F'$ ,  $\partial_\mu \partial^\mu F = k^2 F'' = 0$ ,  $k_\mu A^\mu = 0$ ,  $A_z = 0$  and

$$\eta^z \equiv [\phi_{\mathbf{p}_\perp, \mathcal{E}}^{\text{WKB}}(z)]^{-1} \partial^z \phi_{\mathbf{p}_\perp, \mathcal{E}}^{\text{WKB}}(z) \approx \frac{i}{\hbar} p_z(z). \quad (19)$$

Therefore Eq. (18) becomes

$$\left[ -2i\hbar p_z \frac{d}{dz} + 2\frac{e}{c}(\mathbf{A}_\perp \mathbf{p}_\perp) - \frac{e^2}{c^2} \mathbf{A}_\perp^2 \right] \chi_{\mathbf{p}_\perp, \mathcal{E}}(z) = 0. \quad (20)$$

The integral of this equation is

$$\chi_{\mathbf{p}_\perp, \mathcal{E}}(z) \propto \exp + \frac{i}{\hbar} \int^z \left[ \frac{e^2}{2c^2 p_z} \mathbf{A}_\perp^2 - \frac{e}{c p_z} (\mathbf{A}_\perp \mathbf{p}_\perp) \right] dz', \quad (21)$$

where the integration constant is absorbed into  $\mathcal{C}$  in Eq. (11). The final WKB-solution is then

$$\begin{aligned} \phi_{\mathbf{p}_\perp, \mathcal{E}}^{\text{WKB}}(z) \chi_{\mathbf{p}_\perp, \mathcal{E}}(\varphi) &= \frac{\mathcal{C}}{p_z^{1/2}(z)} e^{iS_{\mathbf{p}_\perp, \mathcal{E}}^{(0)}(z)/\hbar} \chi_{\mathbf{p}_\perp, \mathcal{E}}(\varphi) \\ &= \frac{\mathcal{C}}{p_z^{1/2}(z)} \exp + \frac{i}{\hbar} \int^z \mathcal{P}_z(z') dz', \end{aligned} \quad (22)$$

where

$$\mathcal{P}_z(z) \equiv p_z(z) + \left[ \frac{e^2}{2c^2 p_z} \mathbf{A}_\perp^2 - \frac{e}{c p_z} (\mathbf{A}_\perp \mathbf{p}_\perp) \right]. \quad (23)$$

Between the turning points  $z_- < z < z_+$ , whose positions are illustrated in Fig. 1, the momentum  $p_z(z)$  is imaginary and is useful to define the positive function

$$\kappa_z(z) \equiv \sqrt{p_\perp^2 + m_e^2 c^2 - \frac{1}{c^2} [\mathcal{E} - V(z)]^2} \geq 0, \quad (24)$$

and we define

$$\mathcal{K}_z(z) \equiv \kappa_z(z) - \left[ \frac{e^2}{2c^2 \kappa_z(z)} \mathbf{A}_\perp^2 - \frac{e}{c \kappa_z(z)} (\mathbf{A}_\perp \mathbf{p}_\perp) \right], \quad (25)$$

which depends on the electromagnetic field  $\mathbf{A}_\perp$ . The tunneling wave function in this regime is the linear combination

$$\frac{\mathcal{C}}{2(\kappa_z)^{1/2}} \exp \left[ -\frac{1}{\hbar} \int_{z_-}^z \mathcal{K}_z dz \right] + \frac{\bar{\mathcal{C}}}{2(\kappa_z)^{1/2}} \exp \left[ +\frac{1}{\hbar} \int_{z_-}^z \mathcal{K}_z dz \right]. \quad (26)$$

Outside the turning points, i.e., for  $z < z_-$  and  $z > z_+$ , there exist negative-energy and positive-energy solutions for  $\mathcal{E} < \mathcal{E}_-$  and  $\mathcal{E} > \mathcal{E}_+$  for positive  $p_z$ . On the left-hand side of  $z_-$ , the general solution is a linear combination of an incoming wave running to the right and outgoing wave running to the left:

$$\frac{\mathcal{C}_+}{(p_z)^{1/2}} \exp \left[ \frac{i}{\hbar} \int^z \mathcal{P}_z dz \right] + \frac{\mathcal{C}_-}{(p_z)^{1/2}} \exp \left[ -\frac{i}{\hbar} \int^z \mathcal{P}_z dz \right]. \quad (27)$$

On the right hand of  $z_+$ , there is only an outgoing wave

$$\frac{\mathcal{T}}{(p_z)^{1/2}} \exp \left[ \frac{i}{\hbar} \int_{z_+}^z \mathcal{P}_z dz \right], \quad (28)$$

The connection equations can be solved by

$$\bar{\mathcal{C}} = 0, \quad \mathcal{C}_{\pm} = e^{\pm i\pi/4} \mathcal{C}/2, \quad \mathcal{T} = \mathcal{C}_+ \exp \left[ -\frac{1}{\hbar} \int_{z_-}^{z_+} \mathcal{K}_z dz \right]. \quad (29)$$

The incident flux density is

$$j_z \equiv \frac{\hbar}{2m_e i} [\phi^* \partial_z \phi - (\partial_z \phi^*) \phi] = \frac{p_z}{m_e} \phi^* \phi = \frac{|\mathcal{C}_+|^2}{m_e}, \quad (30)$$

which can be written as

$$j_z(z) = v_z(z) n_-(z), \quad (31)$$

where  $v_z(z) = p_z(z)/m_e$  is the velocity and  $n_-(z) = \phi^*(z)\phi(z)$  the density of the incoming particles. Note that the  $z$ -dependence of  $v_z(z)$  and  $n_-(z)$  cancel each other. By analogy, the outgoing flux density is  $|\mathcal{T}|^2/m_e$ .

## B. Rate of pair production

From the above considerations we obtain for the transmission probability

$$W_{\text{WKB}} \equiv \frac{\text{transmitted flux}}{\text{incident flux}} \quad (32)$$

the simple exponential

$$W_{\text{WKB}}(p_{\perp}, \mathcal{E}, \mathbf{A}_{\perp}) = \exp \left[ -\frac{2}{\hbar} \int_{z_-}^{z_+} \mathcal{K}_z(z') dz' \right]. \quad (33)$$

In order to derive from (32) the total rate of pair production in the electric field we must multiply it with the incident particle flux density at the entrance  $z_-$  of the tunnel. The particle velocity at that point is  $v_z = \partial \mathcal{E} / \partial p_z$ , where the relation between  $\mathcal{E}$  and  $z_-$  is given by Eq. (4):

$$-1 = \frac{\mathcal{E} - V(z_-)}{\sqrt{(cp_{\perp})^2 + m_e^2 c^4}}. \quad (34)$$

This must be multiplied with the particle density which is given by the phase space density  $d^3p/(2\pi\hbar)^3$ . The incident flux density at the tunnel entrance is therefore

$$j_z(z_-) = D_s \int \frac{\partial \mathcal{E}}{\partial p_z} \frac{d^2 p_\perp}{(2\pi\hbar)^2} \frac{dp_z}{2\pi\hbar} = D_s \int \frac{d\mathcal{E}}{2\pi\hbar} \frac{d^2 p_\perp}{(2\pi\hbar)^2}, \quad (35)$$

and the extra factor  $D_s$  is equal to 2 for electrons with two spin orientations [22].

It is useful to change the variable of integration from  $z$  to  $\zeta(z)$  defined by

$$\zeta(p_\perp, \mathcal{E}; z) \equiv \frac{\mathcal{E} - V(z)}{\sqrt{(cp_\perp)^2 + m_e^2 c^4}}, \quad (36)$$

and to introduce the notation for the electric field  $E(p_\perp, \mathcal{E}; \zeta) \equiv E[\bar{z}(p_\perp, \mathcal{E}; \zeta)]$ , where  $\bar{z}(p_\perp, \mathcal{E}; \zeta)$  is the inverse function of (36), the equations in (4) reduce to

$$\zeta_-(p_\perp, \mathcal{E}; z_-) = -1, \quad \zeta_+(p_\perp, \mathcal{E}; z_+) = +1. \quad (37)$$

In terms of the variable  $\zeta$ , the WKB transmission probability (33) can be rewritten as

$$\begin{aligned} W_{\text{WKB}}(p_\perp, \mathcal{E}, \mathbf{A}_\perp) = & \exp \left\{ -\frac{\pi E_c}{E_0} \left[ 1 + \frac{(cp_\perp)^2}{m_e^2 c^4} \right] G(p_\perp, \mathcal{E}) + \right. \\ & \left. + \frac{\pi E_c}{E_0} \left[ \frac{(e\mathbf{A}_\perp)^2}{2m_e^2 c^4} - \frac{e\mathbf{A}_\perp}{m_e^2 c^4} (cp_\perp) \right] H(p_\perp, \mathcal{E}) \right\}, \end{aligned} \quad (38)$$

where the photon amplitude  $\mathbf{A}_\perp$  is independent of space coordinate. Here we have introduced a standard field strength  $E_0$  to make the integral in the exponent dimensionless, which we abbreviate by

$$G(p_\perp, \mathcal{E}) \equiv \frac{2}{\pi} \int_{-1}^1 d\zeta \frac{\sqrt{1-\zeta^2}}{E(p_\perp, \mathcal{E}; \zeta)/E_0} \quad (39)$$

$$H(p_\perp, \mathcal{E}) \equiv \frac{2}{\pi} \int_{-1}^1 d\zeta \frac{(\sqrt{1-\zeta^2})^{-1}}{E(p_\perp, \mathcal{E}; \zeta)/E_0}. \quad (40)$$

The first term in the exponent of (38) is equal to  $2E_c/E_0$ , where

$$E_c \equiv m_e^2 c^3 / e\hbar \quad (41)$$

is the *critical field strength* which creates a pair over two Compton wavelengths  $2\lambda_C = 2\hbar/m_e c$ .

At the semiclassical level, tunneling takes place only if the potential height is larger than  $2m_e c^2$  and for energies  $\mathcal{E}$  for which there are two real turning points  $z_\pm$ . The total tunneling rate is obtained by integrating over all incoming momenta and the total area  $V_\perp = \int dx dy$  of the incoming flux. The WKB-rate per area is

$$\frac{\Gamma_{\text{WKB}}}{V_\perp} = D_s \int \frac{d\mathcal{E}}{2\pi\hbar} \int \frac{d^2 p_\perp}{(2\pi\hbar)^2} W_{\text{WKB}}(p_\perp, \mathcal{E}, \mathbf{A}_\perp). \quad (42)$$



Using the relation following from (34)

$$d\mathcal{E} = eE(z_-)dz_-, \quad (43)$$

we obtain the alternative expression

$$\frac{\Gamma_{\text{WKB}}}{V_{\perp}} = D_s \int \frac{dz_-}{2\pi\hbar} \int \frac{d^2p_{\perp}}{(2\pi\hbar)^2} eE(z_-) W_{\text{WKB}}[p_{\perp}, \mathcal{E}(z_-), \mathbf{A}_{\perp}], \quad (44)$$

where  $\mathcal{E}(z_-)$  is obtained by solving the differential equation (43).

The integral over  $p_{\perp}$  cannot be done exactly. At the semiclassical level, this is fortunately not necessary. Since  $E_c$  is proportional to  $1/\hbar$ , the exponential in (38) restricts the transverse momentum  $p_{\perp}$  to be small of the order of  $\sqrt{\hbar}$ , so that the integral in (44) may be calculated from an expansion of  $G(p_{\perp}, \mathcal{E})$  and  $H(p_{\perp}, \mathcal{E})$  up to the order  $p_{\perp}^2$ :

$$\begin{aligned} G(p_{\perp}, \mathcal{E}) &\simeq \frac{2}{\pi} \int_{-1}^1 d\zeta \frac{\sqrt{1-\zeta^2}}{E(0, \mathcal{E}; \zeta)/E_0} \left[ 1 - \frac{1}{2} \frac{dE(0, \mathcal{E}, \zeta)/d\zeta}{E(0, \mathcal{E}, \zeta)} \zeta \delta + \dots \right] \\ &= G(0, \mathcal{E}) + G_{\delta}(0, \mathcal{E})\delta + \dots, \end{aligned} \quad (45)$$

and

$$\begin{aligned} H(p_{\perp}, \mathcal{E}) &\simeq \frac{2}{\pi} \int_{-1}^1 d\zeta \frac{(\sqrt{1-\zeta^2})^{-1}}{E(0, \mathcal{E}; \zeta)/E_0} \left[ 1 - \frac{1}{2} \frac{dE(0, \mathcal{E}, \zeta)/d\zeta}{E(0, \mathcal{E}, \zeta)} \zeta \delta + \dots \right] \\ &= H(0, \mathcal{E}) + H_{\delta}(0, \mathcal{E})\delta + \dots, \end{aligned} \quad (46)$$

where  $\delta \equiv \delta(p_{\perp}) \equiv (cp_{\perp})^2/(m_e^2 c^4)$ ,

$$\begin{aligned} G_{\delta}(0, \mathcal{E}) &\equiv -\frac{1}{\pi} \int_{-1}^1 d\zeta \frac{\zeta \sqrt{1-\zeta^2}}{E^2(0, \mathcal{E}; \zeta)/E_0} E'(0, \mathcal{E}; \zeta) \\ &= -\frac{1}{2} G(0, \mathcal{E}) + \frac{1}{\pi} \int_{-1}^1 \frac{\zeta^2}{\sqrt{1-\zeta^2}} \frac{d\zeta}{E(0, \mathcal{E}, \zeta)/E_0}, \end{aligned} \quad (47)$$

and

$$\begin{aligned} H_{\delta}(0, \mathcal{E}) &\equiv -\frac{1}{\pi} \int_{-1}^1 d\zeta \frac{\zeta (\sqrt{1-\zeta^2})^{-1}}{E^2(0, \mathcal{E}; \zeta)/E_0} E'(0, \mathcal{E}; \zeta) \\ &= -\frac{1}{2} H(0, \mathcal{E}) - \frac{1}{\pi} \int_{-1}^1 \frac{\zeta^2}{(1-\zeta^2)^{3/2}} \frac{d\zeta}{E(0, \mathcal{E}, \zeta)/E_0}. \end{aligned} \quad (48)$$

We can now perform the integral over  $\mathbf{p}_{\perp}$  in (44) approximately as follows:

$$\begin{aligned} &\int \frac{d^2p_{\perp}}{(2\pi\hbar)^2} \exp \left\{ -\frac{\pi E_c}{E_0} (1+\delta) [G(0, \mathcal{E}) + G_{\delta}(0, \mathcal{E})\delta] + \right. \\ &\quad \left. + \frac{\pi E_c}{E_0} (\tfrac{1}{2}a_{\perp}^2 - a_{\perp}\delta^{1/2}) [H(0, \mathcal{E}) + H_{\delta}(0, \mathcal{E})\delta] \right\} \\ &\approx \frac{m_e^2 c^2}{4\pi\hbar^2} e^{-\frac{\pi E_c}{E_0} [G(0, \mathcal{E}) - \tfrac{1}{2}a_{\perp}^2 H(0, \mathcal{E})]} \int_0^{\infty} d\delta e^{-\pi(E_c/E_0) [\delta \tilde{G}(0, \mathcal{E}) + \delta^{1/2} a_{\perp} H(0, \mathcal{E})]} \\ &= \frac{eE_0}{4\pi^2 \hbar c \tilde{G}(0, \mathcal{E})} e^{-\frac{\pi E_c}{E_0} [G(0, \mathcal{E}) - \tfrac{1}{2}a_{\perp}^2 H(0, \mathcal{E})]} \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}, \end{aligned} \quad (49)$$

where

$$a_{\perp} \equiv \frac{e|\mathbf{A}_{\perp}|}{m_e c^2}, \quad (50)$$

and

$$\tilde{G}(0, \mathcal{E}) \equiv G(0, \mathcal{E}) + G_{\delta}(0, \mathcal{E}) - \frac{1}{2} a_{\perp}^2 H_{\delta}(0, \mathcal{E}) \quad (51)$$

$$\vartheta^2 \equiv \left( \frac{\pi E_c}{E_0} \right) \frac{a_{\perp}^2}{4} \frac{H^2(0, \mathcal{E})}{\tilde{G}(0, \mathcal{E})}. \quad (52)$$

The electric fields  $E(p_{\perp}, \mathcal{E}; \zeta)$  at the tunnel entrance  $z_{-}$  in the prefactor of (44) can be expanded similarly to first order in  $\delta$ . If  $z_{-}^0$  denotes the solutions of (34) at  $p_{\perp} = 0$ , we see that for small  $\delta$ :

$$\Delta z_{-} \equiv z_{-} - z_{-}^0 \approx \frac{m_e c^2}{E(z_{-}^0)} \frac{\delta}{2}. \quad (53)$$

so that

$$E(z_{-}) \simeq E(z_{-}^0) - m_e c^2 \frac{E'(z_{-}^0)}{E(z_{-}^0)} \frac{\delta}{2}. \quad (54)$$

Here the extra term proportional to  $\delta$  can be neglected in the semiclassical limit since it gives a contribution to the prefactor of the order  $\hbar$ . Thus we obtain the WKB-rate (44) of pair production per unit area

$$\begin{aligned} \frac{\Gamma_{\text{WKB}}}{V_{\perp}} &\equiv \int dz \frac{\partial_z \Gamma_{\text{WKB}}(z)}{V_{\perp}} \\ &\simeq D_s \int dz \frac{e^2 E_0 E(z)}{8\pi^3 \hbar^2 c \tilde{G}(0, \mathcal{E}(z))} e^{-\frac{\pi E_c}{E_0} [G(0, \mathcal{E}) - \frac{1}{2} a_{\perp}^2 H(0, \mathcal{E})]} \\ &\cdot \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}, \end{aligned} \quad (55)$$

where  $z$  is short for  $z_{-}^0$ . At this point it is useful to return from the integral  $\int dz_{-} e E(z_{-})$  introduced in (44) to the original energy integral  $\int d\mathcal{E}$  in (42), so that the final result is

$$\begin{aligned} \frac{\Gamma_{\text{WKB}}}{V_{\perp}} &\equiv \int d\mathcal{E} \frac{\partial_{\mathcal{E}} \Gamma_{\text{WKB}}(z)}{V_{\perp}} \\ &\simeq D_s \frac{e E_0}{4\pi^2 \hbar c} \int \frac{d\mathcal{E}}{2\pi \hbar} \frac{1}{\tilde{G}(0, \mathcal{E})} e^{-\frac{\pi E_c}{E_0} [G(0, \mathcal{E}) - \frac{1}{2} a_{\perp}^2 H(0, \mathcal{E})]} \\ &\cdot \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}, \end{aligned} \quad (56)$$

where  $\mathcal{E}$ -integration is over all crossing energy-levels.

These formula can be approximately applied to the 3-dimensional case of electric fields  $\mathbf{E}(x, y, z)$  and potentials  $V(x, y, z)$  at the points  $(x, y, z)$  where the tunneling length  $a \equiv z_{+} - z_{-}$  is much smaller than the variation lengths  $\delta x_{\perp}$  of electric potentials  $V(x, y, z)$  in the  $xy$ -plane,

$$\frac{1}{a} \gg \frac{1}{V} \frac{\delta V}{\delta x_{\perp}}. \quad (57)$$

At these points  $(x, y, z)$ , we can arrange the tunneling path  $dz$  and momentum  $p_z(x, y, z)$  in the direction of electric field, corresponding perpendicular area  $d^2V_\perp \equiv dxdy$  for incident flux and perpendicular momentum  $\mathbf{p}_\perp$ . It is then approximately reduced to a one-dimensional problem in the region of size  $\mathcal{O}(a)$  around these points. The surfaces  $z_-(x_-, y_-, \mathcal{E})$  and  $z_+ = (x_+, y_+, \mathcal{E})$  associated with the classical turning points are determined by Eqs. (36) and Eqs. (37) for a given energy  $\mathcal{E}$ . The WKB-rate of pair production (55) can then be expressed as an volume integral over the rate density per volume element

$$\Gamma_{\text{WKB}} = \int dxdydz \frac{d^3\Gamma_{\text{WKB}}}{dx dy dz} = \int dt dxdydz \frac{d^4N_{\text{WKB}}}{dt dx dy dz}. \quad (58)$$

On the right-hand side we have found it useful to rewrite the rate  $\Gamma_{\text{WKB}}$  as the time derivative of the number of pair creation events  $dN_{\text{WKB}}/dt$ , so that we obtain an event density in four-space

$$\begin{aligned} \frac{d^4N_{\text{WKB}}}{dt dx dy dz} \approx & D_s \frac{e^2 E_0 E(z)}{8\pi^3 \hbar \tilde{G}(0, \mathcal{E}(z))} e^{-\frac{\pi E_c}{E_0} [G(0, \mathcal{E}) - \frac{1}{2} a_\perp^2 H(0, \mathcal{E})]} \\ & \cdot \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}, \end{aligned} \quad (59)$$

Here  $x, y$  and  $z$  are related by the function  $z = z_-(x, y, \mathcal{E})$  which is obtained by solving (43).

It is now useful to observe that the left-hand side of (59) is a Lorentz-invariant quantity. In addition, it is symmetric under the exchange of time and  $z$ , and this symmetry will be exploited in the next section to relate pair production processes in a  $z$ -dependent electric field  $E(z)$  to those in a time-dependent field  $E(t)$ .

Attempts to go beyond the WKB results (55) or (56) require a great amount of work. Corrections will come from three sources:

- I from the higher terms of order in  $(\hbar)^n$  with  $n > 1$  in the the expansion (15) solving the Riccati equation (13).
- II from the higher terms of the perturbative evaluation of the integral over  $\mathbf{p}_\perp$  in Eqs. (42) or (44) when going beyond the Gaussian approximation.
- III from perturbative corrections to the Gaussian energy integral (56) or the corresponding  $z$ -integral (55).

All these corrections contribute terms of higher order in  $\hbar$ .

### C. Including a Smoothly Varying $\mathbf{B}(z)$ -Field Parallel to $\mathbf{E}(z)$

The above results can easily be extended to allow for the presence of a constant magnetic field  $\mathbf{B}$  parallel to  $\mathbf{E}(z)$ . Then the wave function factorizes into a Landau state and a spinor function first calculated by Sauter [1]. In the WKB approximation, the energy spectrum is still given by Eq. (2), but the squared transverse momenta  $p_\perp^2$  is quantized and must be replaced by discrete values corresponding to the Landau energy levels. From the known nonrelativistic levels for the Hamiltonian  $p_\perp^2/2m_e$  we extract immediately the replacements

$$c^2 p_\perp^2 = 2m_e c^2 \times \left( \frac{p_\perp^2}{2m_e} \right) \longrightarrow 2m_e c^2 \times \left[ \hbar \omega_L \left( n + \frac{1}{2} + g\sigma \right) \right], \quad n = 0, 1, 2, \dots, \quad (60)$$

where  $g = 2 + \alpha/\pi + \dots$  is the anomalous magnetic moment of the electron,  $\omega_L = eB/m_e c$  the Landau frequency, with  $\sigma = \pm 1/2$  for spin-1/2 and  $\sigma = 0$  for spin-0, which are eigenvalues of the Pauli matrix  $\sigma_z$ . The quantum number  $n$  characterizing the Landau levels counts the levels of the harmonic oscillations in the plane orthogonal to the  $z$ -direction. Apart from the replacement (60), the WKB calculations remain the same. Thus we must only replace the integration over the transverse momenta  $\int d^2 p_\perp / (2\pi\hbar)^2$  in Eq. (49) by the sum over all Landau levels with the degeneracy  $eB/(2\pi\hbar c)$ . Thus, the right-hand side becomes

$$\frac{eB}{2\pi\hbar c} e^{-\pi(E_c/E_0)[G(0,\mathcal{E}) - \frac{1}{2}a_\perp^2 H(0,\mathcal{E})]} \sum_{n,\sigma} e^{-\pi(B/E_0)[(n+1/2+g\sigma)\tilde{G}(0,\mathcal{E}) + (n+1/2+g\sigma)^{1/2} a_\perp H(0,\mathcal{E})]}. \quad (61)$$

The approximate result is, for spin-0 and spin-1/2:

$$\frac{eE_0}{4\pi^2\hbar c\tilde{G}(0,\mathcal{E})} e^{-\pi(E_c/E_0)[G(0,\mathcal{E}) - \frac{1}{2}a_\perp^2 H(0,\mathcal{E})]} f_{0,1/2}(B\tilde{G}(0,\mathcal{E})/E_0) \quad (62)$$

where

$$f_0(x) \equiv \frac{\pi x}{\sinh \pi x}, \quad f_{1/2}(x) \equiv 2 \frac{\pi x}{\sinh \pi x} \cosh \frac{\pi g x}{2} \quad (63)$$

In the limit  $B \rightarrow 0$ , Eq. (63) reduces to Eq. (49).

The result remains approximately valid if the magnetic field has a smooth  $z$ -dependence varying little over a Compton wavelength  $\lambda_e$ .

In the following we shall focus on nonuniform electric fields without a magnetic field.

### III. TIME-DEPENDENT ELECTRIC FIELDS

The above semiclassical considerations can be applied with little change to the different physical situation in which the electric field along the  $z$ -direction depends only on time rather than  $z$ . Instead

of the time  $t$  itself we shall prefer working with the zeroth length coordinate  $x_0 = ct$ , as usual in relativistic calculations. As an intermediate step consider for a moment a vector potential

$$A_\mu = (A_0(z), 0, 0, A_z(x_0)), \quad (64)$$

with the electric field

$$E = -\partial_z A_0(z) - \partial_0 A_z(x_0), \quad x_0 \equiv ct. \quad (65)$$

The associated Klein-Gordon equation (7) reads

$$\left\{ \left[ i\hbar\partial_0 + \frac{e}{c}A_0(z) \right]^2 + \hbar^2\partial_{\mathbf{x}_\perp}^2 - \left[ i\hbar\partial_z + \frac{e}{c}A_z(x_0) \right]^2 - m_e^2c^2 + 2i\hbar\frac{e}{c}(A_\mu\partial^\mu) + \frac{e^2}{c^2}\mathbf{A}_\perp^2 \right\} \phi(x) = 0. \quad (66)$$

The previous discussion was valid under the assumption  $A_z(x_0) = 0$ , in which case the ansatz (9) led to the field equation (10). For the present discussion it is useful to write the ansatz as

$$\phi(x) = e^{-ip_0x_0/\hbar} e^{i\mathbf{p}_\perp\mathbf{x}_\perp/\hbar} \phi_{\mathbf{p}_\perp, p_0}(z) \chi_{\mathbf{p}_\perp, p_0}(z) \quad (67)$$

with  $p_0 = \mathcal{E}/c$ , and Eq. (10) in the form

$$\left\{ \frac{1}{c^2} \left[ \mathcal{E} - e \int^z dz' E(z') \right]^2 - p_\perp^2 - m_e^2c^2 + \hbar^2 \frac{d^2}{dz^2} - 2\frac{e}{c}(\mathbf{A}_\perp \mathbf{p}_\perp) + \frac{e^2}{c^2} \mathbf{A}_\perp^2 \right\} \phi_{\mathbf{p}_\perp, p_0}(z) \chi_{\mathbf{p}_\perp, p_0}(z) = 0 \quad (68)$$

Now we assume the electric field to depend only on  $x_0 = ct$ . Then the ansatz:

$$\phi(x) = e^{ip_z z/\hbar} e^{i\mathbf{p}_\perp\mathbf{x}_\perp/\hbar} \phi_{\mathbf{p}_\perp, p_z}(x_0) \chi_{\mathbf{p}_\perp, p_z}(x_0) \quad (69)$$

leads to the field equation

$$\left\{ -\hbar^2 \frac{d^2}{dx_0^2} - p_\perp^2 - m_e^2c^2 - \left[ -p_z - \frac{e}{c} \int^{x_0} dx'_0 E(x'_0) \right]^2 - 2\frac{e}{c}(\mathbf{A}_\perp \mathbf{p}_\perp) + \frac{e^2}{c^2} \mathbf{A}_\perp^2 \right\} \phi_{\mathbf{p}_\perp, p_z}(x_0) \chi_{\mathbf{p}_\perp, p_z}(x_0) = 0 \quad (70)$$

If we compare Eq. (70) with (68) we realize that one arises from the other by interchanging

$$z \leftrightarrow x_0, \quad p_\perp \rightarrow ip_\perp, \quad c \rightarrow ic, \quad E \rightarrow -iE. \quad (71)$$

With these exchanges we may easily calculate the decay rate of the vacuum caused by a time-dependent electric field  $E(x_0)$  using the above-derived formulas.

#### IV. APPLICATIONS

In this section, we apply formulas (56) or (55) to two typical external field configurations

### A. Step-like electric field

First we check our result for the original case of a constant electric field  $E(z) \equiv eE_0$  where the potential energy is the linear function  $V(z) = -eE_0z$ . Here the functions (39) and (40) become trivial

$$G(0, \mathcal{E}) = \frac{2}{\pi} \int_{-1}^1 d\zeta \sqrt{1 - \zeta^2} = 1, \quad G_\delta(0, \mathcal{E}) = 0, \quad (72)$$

$$H(0, \mathcal{E}) = \frac{2}{\pi} \int_{-1}^1 d\zeta (\sqrt{1 - \zeta^2})^{-1} = 1, \quad H_\delta(0, \mathcal{E}) = -1, \quad (73)$$

$$\tilde{G}(0, \mathcal{E}) = 1 + a_\perp^2/2 \quad (74)$$

$$\vartheta^2 = \left( \frac{\pi E_c}{E_0} \right) \frac{a_\perp^2}{4} \frac{1}{1 + a_\perp^2/2}. \quad (75)$$

which is independent of  $\mathcal{E}$  (or  $z_-$ ). The WKB-rate for pair-production per unit time and volume is found from Eq. (55) to be

$$\frac{\Gamma_{\text{WKB}}^{\text{EH}}}{V} \simeq D_s \frac{e^2 E_0^2}{8\pi^3 \hbar^2 c} \frac{1}{1 + a_\perp^2/2} e^{-\frac{\pi E_c}{E_0}(1 - a_\perp^2/2)} \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}. \quad (76)$$

where  $V \equiv dz_- V_\perp$ . We find that photon field amplitude squared  $a_\perp^2$  (50) gives rise to an exponential factor of enhancement  $e^{(\pi E_c/E_0)(a_\perp^2/2)}$ , we will turn to this point later. For  $a_\perp \rightarrow 0$  and  $\vartheta \rightarrow 0$ , Eq. (76) goes to the correct expression found by Sauter [1], Heisenberg and Euler [2], and by Schwinger [3].

### B. Sauter electric field

Let us now consider the nontrivial Sauter electric field concentrated to a thin slab in the  $xy$ -plane with a width  $\ell$  in the  $z$ -direction. A field of this type can be produced, e.g., between two opposite charged conducting plates. The electric field  $E(z)\hat{\mathbf{z}}$  in the  $z$ -direction and the associated potential energy  $V(z)$  are given by

$$E(z) = E_0 / \cosh^2(z/\ell), \quad V(z) = -\sigma m_e c^2 \tanh(z/\ell), \quad (77)$$

where

$$\sigma \equiv eE_0\ell/m_e c^2 = (\ell/\lambda_C)(E_0/E_c). \quad (78)$$

From now on we shall use natural units in which energies are measured in units of  $m_e c^2$ . Figure 1 shows the positive and negative-energy spectra  $\mathcal{E}_\pm(z)$  of Eq. (2) for  $p_z = p_\perp = 0$  to show the energy-gap and energy-level crossings. From Eq. (4) we find the classical turning points

$$z_\pm = \ell \operatorname{arctanh} \frac{\mathcal{E} \pm \sqrt{1 + \delta}}{\sigma} = \frac{\ell}{2} \ln \frac{\sigma + \mathcal{E} \pm \sqrt{1 + \delta}}{\sigma - \mathcal{E} \mp \sqrt{1 + \delta}}. \quad (79)$$

Tunneling is possible for all energies satisfying

$$-\sqrt{1+\delta} + \sigma \geq \mathcal{E} \geq \sqrt{1+\delta} - \sigma, \quad (80)$$

for the strength parameter  $\sigma > \sqrt{1+\delta}$ .

We may invert Eq. (36) to find the relation between  $\zeta$  and  $z$ :

$$z = z(p_\perp, \mathcal{E}; \zeta) = \ell \operatorname{arctanh} \frac{\mathcal{E} + \zeta \sqrt{1+\delta}}{\sigma} = \frac{\ell}{2} \ln \frac{\sigma + \mathcal{E} + \zeta \sqrt{1+\delta}}{\sigma - \mathcal{E} - \zeta \sqrt{1+\delta}}. \quad (81)$$

In terms of the function  $z(p_\perp, \mathcal{E}; \zeta)$ , the equation (79). reads simply  $z_\pm = z(p_\perp, \mathcal{E}; \pm 1)$ .

In Ref. [12], inserting (81) into the equation for  $E(z)$  in Eq. (77), we obtain

$$E(z) = E_0 \left[ 1 - \left( \frac{\zeta \sqrt{1+\delta} - \mathcal{E}}{\sigma} \right)^2 \right] \equiv E(p_\perp, \mathcal{E}; \zeta), \quad (82)$$

and calculate  $G(0, \mathcal{E})$  and  $G_\delta(0, \mathcal{E})$  of Eqs. (39), (45) and (47):

$$G(0, \mathcal{E}) = 2\sigma^2 - \sigma \left[ (\sigma - \mathcal{E})^2 - 1 \right]^{1/2} - \sigma \left[ (\sigma + \mathcal{E})^2 - 1 \right]^{1/2}, \quad (83)$$

and

$$G(0, \mathcal{E}) + G_\delta(0, \mathcal{E}) = \frac{\sigma}{2} \left\{ \left[ (\sigma - \mathcal{E})^2 - 1 \right]^{-1/2} + \left[ (\sigma + \mathcal{E})^2 - 1 \right]^{-1/2} \right\}. \quad (84)$$

The integral over all energy-level crossings permitted by the energy inequality (80) is dominated by the region around  $\mathcal{E} \sim 0$ , where the tunneling length is shortest [see Fig. 1] and tunneling probability is largest. Both functions  $G(0, \mathcal{E})$  and  $G_\delta(0, \mathcal{E})$  have a symmetric peak at  $\mathcal{E} = 0$ . Around the peak they can be expanded in powers of  $\mathcal{E}$  as

$$\begin{aligned} G(0, \mathcal{E}) &= 2[\sigma^2 - \sigma(\sigma^2 - 1)^{1/2}] + \frac{\sigma}{(\sigma^2 - 1)^{3/2}} \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \\ &= G_0(\sigma) + \frac{1}{2} G_2(\sigma) \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4), \end{aligned} \quad (85)$$

and

$$\begin{aligned} G(0, \mathcal{E}) + G_\delta(0, \mathcal{E}) &= \frac{\sigma}{(\sigma^2 - 1)^{1/2}} + \frac{1}{2} \frac{(1 + 2\sigma^2)}{(\sigma^2 - 1)^{5/2}} \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4) \\ &= \overline{G}_0(\sigma) + \frac{1}{2} \overline{G}_2(\sigma) \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4). \end{aligned} \quad (86)$$

The exponential  $e^{-\pi G(0, \mathcal{E}) E_c / E_0}$  has a Gaussian peak around  $\mathcal{E} = 0$  whose width is of the order of  $1/E_c$ , which in physical units is a first-order term in  $\hbar$ . This implies that in the semiclassical limit, we may perform only a Gaussian integral and neglect the  $\mathcal{E}$ -dependence of the prefactor in (56).

Analogously, the function  $H(0, \mathcal{E})$  of Eq. (46) is given by

$$\begin{aligned}
H(0, \mathcal{E}) &= \frac{2}{\pi} \int_{-1}^1 d\zeta \frac{(\sqrt{1-\zeta^2})^{-1}}{1 - \left(\frac{\zeta-\mathcal{E}}{\sigma}\right)^2} \\
&= \frac{2\sigma}{(1+\sigma^2)^{1/2}} + \frac{\sigma(1-2\sigma^2)}{(1+\sigma^2)^{5/2}} \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4), \\
&\equiv \overline{H}_0(\sigma) + \overline{H}_2(\sigma) \mathcal{E}^2 + \mathcal{O}(\mathcal{E}^4),
\end{aligned} \tag{87}$$

and  $H_\delta(0, \mathcal{E})$  Eq. (48)

$$\begin{aligned}
H_\delta(0, \mathcal{E}) &= -\frac{1}{2} H(0, \mathcal{E}) - \frac{1}{\pi} \int_{-1}^1 d\zeta \frac{\zeta^2}{(1-\zeta^2)^{3/2}} \frac{d\zeta}{1 - \left(\frac{\zeta-\mathcal{E}}{\sigma}\right)^2} \\
&= -\frac{3}{2} [\overline{H}_0(\sigma) + \overline{H}_2(\sigma) \mathcal{E}^2] + \mathcal{O}(\mathcal{E}^4).
\end{aligned} \tag{88}$$

The equations (51) and (52) become

$$\begin{aligned}
\tilde{G}(0, \mathcal{E}) &= \overline{G}_0(\sigma) + \frac{1}{2} \overline{G}_2(\sigma) \mathcal{E}^2 \\
&+ \frac{3}{4} a_\perp^2 [\overline{H}_0(\sigma) + \overline{H}_2(\sigma) \mathcal{E}^2] + \mathcal{O}(\mathcal{E}^4)
\end{aligned} \tag{89}$$

$$\begin{aligned}
\vartheta^2 &\equiv \left( \frac{\pi E_c}{E_0} \right) \frac{a_\perp^2}{4} \frac{H^2(0, \mathcal{E})}{\tilde{G}(0, \mathcal{E})}, \\
&= \left( \frac{\pi E_c}{E_0} \right) \frac{a_\perp^2}{4} \frac{\overline{H}_0^2(\sigma)}{\overline{G}_0(\sigma) + \frac{3}{4} a_\perp^2 \overline{H}_0(\sigma)} + \mathcal{O}(\mathcal{E}^2).
\end{aligned} \tag{90}$$

Recalling that  $\mathcal{E}$  in this section is in natural units with  $m_e c^2 = 1$ , we must replace  $\int d\mathcal{E}$  in the pair-production rate (56) by  $m_e c^2 \int d\mathcal{E}$  and can perform the integral over  $\mathcal{E}$  approximately as follows

$$\begin{aligned}
\frac{\Gamma_{\text{WKB}}}{V_\perp} &\simeq D_s \frac{e E_0 m_e c^2}{4\pi^2 \hbar c} \frac{1}{\overline{G}_0 + \frac{3}{4} a_\perp^2 \overline{H}_0} \\
&\times e^{-\pi(E_c/E_0)(\overline{G}_0 - \frac{1}{2} a_\perp^2 \overline{H}_0)} \int \frac{d\mathcal{E}}{2\pi \hbar} e^{-\pi(E_c/E_0)(\overline{G}_2 - a_\perp^2 \overline{H}_2) \mathcal{E}^2/2} \\
&\approx D_s \frac{e E_0}{4\pi^2 \hbar c} \frac{1}{\overline{G}_0 + \frac{3}{4} a_\perp^2 \overline{H}_0} \frac{e^{-\pi(E_c/E_0)(\overline{G}_0 - \frac{1}{2} a_\perp^2 \overline{H}_0)}}{2\pi \hbar [(\overline{G}_2 - a_\perp^2 \overline{H}_2) E_c / 2 E_0]^{1/2}}.
\end{aligned} \tag{91}$$

For convenience, we have extended the limits of integration over  $\mathcal{E}$  from the interval  $(-1+\sigma, 1-\sigma)$  to  $(-\infty, \infty)$ . This introduces exponentially small errors and can be ignored.

## V. COHERENT AND THERMAL STATES OF PHOTON FIELDS

Now we turn to the calculation of the average of  $a_\perp^2 = (e\mathbf{A}_\perp)^2/(m_e^2 c^4)$  [Eq. (50)] of photon fields  $A_\mu$  in coherent monochromatic state and thermal state.



The pair-production rate (59) depends on the transverse amplitude  $\mathbf{A}_\perp(t, z)$  (5) of monochromatic electromagnetic wave  $\omega = |\mathbf{k}| = k_z$ , we need to take the average

$$\left\langle \frac{d^4 N_{\text{WKB}}}{dt dx dy dz} \right\rangle$$

over amplitudes  $\mathbf{A}_\perp$ . Using the convexity inequality [23]

$$\langle e^{\mathbf{A}_\perp^2} \rangle \geq e^{\langle \mathbf{A}_\perp^2 \rangle}, \quad (92)$$

we can obtain the low bound in the pair-production rate Eq. (59).

For the case of monochromatic electromagnetic wave (6) with its transversed amplitude  $\mathbf{A}_\perp(k)$ ,  $\mathbf{A}_\perp^*(k) = \mathbf{A}_\perp(-k)$ , and the corresponding electric component

$$\mathbf{E}_\perp(t, z) = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}_\perp(t, z) = -i \frac{\omega}{c} \mathbf{A}_\perp(t, z), \quad (93)$$

and the maximal amplitude  $E_{\text{peak}} = \omega |\mathbf{A}_\perp(k)|/c$ . In a monochromatical state of laser photon, averaging over one period  $\mathcal{T} = 2\pi/\omega$ , we have

$$\langle \mathbf{A}_\perp^2(t, z) \rangle = \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} dt \mathbf{A}_\perp^*(t, z) \mathbf{A}_\perp(t, z) = \mathbf{A}_\perp^*(k) \mathbf{A}_\perp(k), \quad (94)$$

and

$$\frac{1}{2} \langle a_\perp^2 \rangle = \frac{1}{2} \left\langle \frac{(e \mathbf{A}_\perp)^2}{m_e^2 c^4} \right\rangle = \frac{1}{2} \frac{e^2}{m_e^2 c^4} \mathbf{A}_\perp^*(k) \mathbf{A}_\perp(k) = \frac{1}{2} \left( \frac{m_e c^2}{\hbar \omega} \right)^2 \left( \frac{E_{\text{peak}}}{E_c} \right)^2. \quad (95)$$

This shows the range of  $E_{\text{peak}}$  and  $\hbar \omega$  in order to have a significant enhancement.

Let us now consider a general gauge field ( $\hbar = c = 1$ ),

$$\mathbf{A}_\perp(x) = \int \frac{d^4 k}{(2\pi)^4} \delta_+(k^2) \mathbf{A}_\perp(k) e^{ikx} = \int \frac{d^3 k}{(2\pi)^3 \omega_k} \delta_+(k^2) \mathbf{A}_\perp(k) e^{ikx}, \quad (96)$$

where  $\omega_k$  is the dispersion relation of electromagnetic fields  $\mathbf{A}_\perp(x)$ . Averaging over space-time, we have

$$\begin{aligned} \langle \mathbf{A}_\perp^*(x) \mathbf{A}_\perp(x) \rangle &= \int d^4 x \mathbf{A}_\perp^*(x) \mathbf{A}_\perp(x) = \int \frac{d^4 k}{(2\pi)^4} \mathbf{A}_\perp^*(k) \mathbf{A}_\perp(k) \\ &= \int \frac{d^3 k}{(2\pi)^3 \omega_k} \mathbf{A}_\perp^*(k) \mathbf{A}_\perp(k), \end{aligned} \quad (97)$$

where  $\omega_k = |k|$ . Here  $\mathbf{A}_\perp^*(k) \mathbf{A}_\perp(k)$  is the number density of photons in the momentum state  $k$ . In the case of thermal photons at temperature  $T$ , it is equal to

$$f_\gamma(k) = \frac{1}{e^{\omega_k/T} - 1}. \quad (98)$$

Here and in the sequel  $T$  will be measured in natural units in which the Boltzmann constant  $k_B$  is equal to unity. Hence the finite- $T$  version of Eq. (95) is

$$\begin{aligned} \frac{1}{2}\langle a_{\perp}^2 \rangle &= \frac{2\alpha}{2m_e^2 c^4} \int \frac{d^3 k}{(2\pi)^3 \omega_k} f_{\gamma}(k) \\ &= \frac{2\alpha}{2m_e^2 c^4} \int \frac{d^3 k}{(2\pi)^3 \omega_k} \frac{1}{e^{\omega_k/T} - 1} = \frac{\alpha}{12} \left( \frac{T}{m_e c^2} \right)^2. \end{aligned} \quad (99)$$

This shows that the enhancement is very small for  $T \leq m_e c^2$ .

## VI. THE PRESENCE OF ELECTRONS WITH TEMPERATURE AND CHEMICAL POTENTIAL

In the previous section (V), we have considered the pair-production rate in the case of thermal photons at temperature  $T$  (98). We further consider the pair-production rate in the gas of thermal electrons at the same temperature  $T$ . These thermal electrons are in the Fermi distribution,

$$f_e(\mathcal{E}_e, \mu_e, T) = \frac{1}{e^{(\mathcal{E}_e - \mu_e)/T} + 1}, \quad (100)$$

where the electron energy-level

$$\mathcal{E}_e = [(cp_e)^2 + m_e^2 c^2]^{1/2}, \quad (101)$$

the associated electron number-density is

$$n_e(m_e, \mu_e, T) = 2 \int \frac{d^3 p_e}{(2\pi\hbar)^3} \frac{1}{e^{(\mathcal{E}_e - \mu_e)/T} + 1}, \quad (102)$$

and the chemical potential  $\mu_e > 0$  that is related to the total number  $\mathcal{N}_e$  of electrons. The rate of pair-production is given by

$$\frac{\Gamma_{\text{WKB}}}{V_{\perp}} = D_s \int \frac{d\mathcal{E}}{2\pi\hbar} [1 - f_e(\mathcal{E}, \mu_e, T)] \int \frac{d^2 p_{\perp}}{(2\pi\hbar)^2} W_{\text{WKB}}(p_{\perp}, \mathcal{E}, A), \quad (103)$$

where “ $\mathcal{E}$ ” energy-level-crossings for pair-productions, and the inserted Pauli-blocking factor  $[1 - f_e(\mathcal{E}, \mu_e, T)]_{\mathcal{E}=\mathcal{E}_e}$  gives a probability whether the energy-level  $\mathcal{E}_e = \mathcal{E}$  is occupied. This limits the phase-space permitted by available energy-level-crossings “ $\mathcal{E}$ ” for pair-productions.

In the case of constant electric fields (76), the pair-production probability  $W_{\text{WKB}}(p_{\perp}, \mathcal{E}, A)$  is independent of the energy-crossing-level “ $\mathcal{E}$ ”. We first consider the constant electric field  $E_0$  confined within the finite “box” region  $[-\ell/2, \ell/2]$  in the  $\hat{z}$ -direction, and the range of energy-level crossing is  $[\mathcal{E}_+, \mathcal{E}_-]$  and  $\mathcal{E}_- > \mathcal{E}_+$ , where

$$\mathcal{E}_{\pm} = \mp e E_0 \ell / 2, \quad (104)$$

$\mathcal{E}_- > 0$  and  $\mathcal{E}_+ < 0$  (Fig. 1 presents a similar case). Furthermore, we assume that in this finite “box” region there are electrons whose density is given by  $n_e(\mu_e, T)$  [see Eq. (102)]. We can calculate the pair-production rate by integrating over energy-crossing-levels

$$\int_{\mathcal{E}_+}^{\mathcal{E}_-} \frac{d\mathcal{E}}{2\pi\hbar} [1 - f_e(\mathcal{E}, \mu_e, T)] = \frac{T}{2\pi\hbar} \ln \left( \frac{e^{\mathcal{E}_-/T} + e^{\mu_e/T}}{e^{\mathcal{E}_+/T} + e^{\mu_e/T}} \right) \quad (105)$$

where  $\mathcal{E}_- \geq \mu_e \geq \mathcal{E}_+$ . As a result, the pair-production rate per area (76) is modified as follows

$$\begin{aligned} \frac{\Gamma_{\text{WKB}}^{\text{EH}}}{V_{\perp}} &\simeq D_s \frac{eE_0 T}{8\pi^3 \hbar^2 c} \ln \left( \frac{e^{\mathcal{E}_-/T} + e^{\mu_e/T}}{e^{\mathcal{E}_+/T} + e^{\mu_e/T}} \right) \\ &\times \frac{1}{1 + \langle a_{\perp}^2 \rangle / 2} e^{-\frac{\pi E_c}{E_0} (1 - \langle a_{\perp}^2 \rangle / 2)} \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}, \end{aligned} \quad (106)$$

where  $\vartheta = \vartheta(\langle a_{\perp}^2 \rangle)$  [see Eq. (75)]. Eq. (105) plays a suppression factor in the pair-production rate (106).

In the low-temperature limit  $T/\mu_e \ll 1$  and electron chemical potential goes to its Fermi energy  $E_F$ ,  $\mu_e \rightarrow E_F$  the leading order of Eq. (105) is given by

$$\frac{\mathcal{E}_- - \mu_e}{2\pi\hbar} \approx \frac{eE_0\ell/2 - E_F}{2\pi\hbar}. \quad (107)$$

When  $\mathcal{E}_- - \mu_e = eE_0\ell/2 - E_F = 0$ , which means all energy crossing-levels for pair-productions are fully filled by electrons, it leads to a complete Pauli-blocking, and vanishing of pair-production rate (106). In the high-temperature limit  $T/\mu_e \gg 1$ , the leading order of Eq. (105) is given by

$$\frac{eE_0\ell}{2\pi\hbar}, \quad (108)$$

and Eq. (106) correctly goes back to the expression (76).

In the case of Sauter electric field (77) for the semiclassical limit, neglecting  $\mathcal{E}$ -dependence in the prefactor, we consider the following Gaussian  $\mathcal{E}$ -integration (see Fig. 1)

$$\int_{1-\sigma}^{\sigma-1} \frac{d\mathcal{E}}{2\pi\hbar} e^{-\pi(E_c/E_0)(\overline{G}_2 - a_{\perp}^2 \overline{H}_2) \mathcal{E}^2/2} [1 - f_e(\mathcal{E}, \mu_e, T)] \approx \frac{T}{2\pi\hbar} \ln \left( \frac{e^{\mathcal{E}_+/T} + e^{\mu_e/T}}{e^{\mathcal{E}_-/T} + e^{\mu_e/T}} \right), \quad (109)$$

where the exponential factor plays a cutoff at

$$\mathcal{E}_{\pm} = \pm \left( \frac{2}{\pi} \right)^{1/2} (E_c/E_0)^{-1/2} (\overline{G}_2 - a_{\perp}^2 \overline{H}_2)^{-1/2}, \quad (110)$$

$\sigma \geq \mathcal{E}_+$  and  $\mathcal{E}_- \geq -\sigma$ . To see the Pauli-blocking effect, we further neglect  $\mathcal{E}$ -dependence of the exponential factor of Eq. (109), and assume the maximal pair-production probability at  $\mathcal{E} = 0$ , and Eq. (109) approximately becomes

$$\int_{1-\sigma}^{\sigma-1} \frac{d\mathcal{E}}{2\pi\hbar} [1 - f_e(\mathcal{E}, \mu_e, T)] = \frac{T}{2\pi\hbar} \ln \left( \frac{e^{(\sigma-1)/T} + e^{\mu_e/T}}{e^{(1-\sigma)/T} + e^{\mu_e/T}} \right). \quad (111)$$

In consequence, the pair-production rate in the Sauter field Eq. (91) multiplied by this expression factor (111). We have the same discussions on the high- and low- temperature limits by the replacement  $\sigma - 1 \Rightarrow eE\ell/2$ .

## VII. THE PRESENCE OF BOSONS WITH TEMPERATURE AND CHEMICAL POTENTIAL

In the previous section (VI), we have considered the suppression of pair-production rate in the presence of thermal electrons at temperature  $T$  and chemical potential  $\mu_e$ . Considering charged bosons  $\phi$ , we further consider the enhancement of pair-production rate in the gas of thermal bosons at the same temperature  $T$ . These bosons are in the Bose-Einstein distribution

$$f_\phi(\mathcal{E}_\phi, \mu_\phi, T) = \frac{1}{e^{(\mathcal{E}_\phi - \mu_\phi)/T} - 1}, \quad (112)$$

where the boson energy-level

$$\mathcal{E}_\phi = [(cp_\phi)^2 + m_\phi^2 c^2]^{1/2}, \quad (113)$$

the associated boson number-density

$$n_\phi(m_\phi, \mu_\phi, T) = 2 \int \frac{d^3 p_\phi}{(2\pi\hbar)^3} \frac{1}{e^{(\mathcal{E}_\phi - \mu_\phi)/T} - 1}, \quad (114)$$

where  $\mathcal{E}_\phi > \mu_\phi$  and the chemical potential  $\mu_\phi$  is related to the total number  $\mathcal{N}_\phi$  of bosons. The rate of pair-production is given by

$$\frac{\Gamma_{\text{WKB}}}{V_\perp} = D_s \int \frac{d\mathcal{E}}{2\pi\hbar} [1 + f_\phi(\mathcal{E}, \mu_\phi, T)] \int \frac{d^2 p_\perp}{(2\pi\hbar)^2} W_{\text{WKB}}(p_\perp, \mathcal{E}, A), \quad (115)$$

where “ $\mathcal{E}$ ” energy-level-crossings for pair-productions, and the inserted Bose-Einstein enhancement factor  $[1 + f_\phi(\mathcal{E}, \mu_\phi, T)]_{\mathcal{E}=\mathcal{E}_\phi}$  gives a probability that more particles can occupy the energy-level  $\mathcal{E}_\phi = \mathcal{E}$ . This enlarges the phase-space permitted by available energy-level-crossings “ $\mathcal{E}$ ” for pair-productions.

In the case of constant electric fields (76), the pair-production probability  $W_{\text{WKB}}(p_\perp, \mathcal{E}, A)$  is independent of the energy-crossing-level “ $\mathcal{E}$ ”. We first consider the constant electric field  $E_0$  confined within the finite “box” region  $[-\ell/2, \ell/2]$  in the  $\hat{z}$ -direction, and the range of energy-level crossing is  $[\mathcal{E}_+, \mathcal{E}_-]$  and  $\mathcal{E}_- > \mathcal{E}_+$ , see Eq. (104). Furthermore, we assume that in this finite “box” region there are bosons whose density is given by  $n_\phi$  (114). We can calculate the pair-production rate by integrating over energy-crossing-levels

$$\int_{\mathcal{E}_+}^{\mathcal{E}_-} \frac{d\mathcal{E}}{2\pi\hbar} [1 + f_\phi(\mathcal{E}, \mu_\phi, T)] = \frac{1}{\pi\hbar} (\mathcal{E}_- - \mathcal{E}_+) - \frac{T}{2\pi\hbar} \ln \left( \frac{e^{\mathcal{E}_-/T} - e^{\mu_\phi/T}}{e^{\mathcal{E}_+/T} - e^{\mu_\phi/T}} \right) \quad (116)$$

where  $(\mathcal{E}_- - \mathcal{E}_+) = eE_0\ell$  and  $\mathcal{E}_- \geq \mathcal{E}_+ \geq \mu_\phi$ . As a result, the pair-production rate per area (76) is modified as follows

$$\begin{aligned} \frac{\Gamma_{\text{WKB}}^{\text{EH}}}{V_\perp} &\simeq D_s \frac{eE_0 T}{8\pi^3 \hbar^2 c} \left[ \frac{2}{T} (\mathcal{E}_- - \mathcal{E}_+) - \ln \left( \frac{e^{\mathcal{E}_-/T} - e^{\mu_\phi/T}}{e^{\mathcal{E}_+/T} - e^{\mu_\phi/T}} \right) \right] \\ &\times \frac{1}{1 + \langle a_\perp^2 \rangle / 2} e^{-\frac{\pi E_c}{E_0} (1 - \langle a_\perp^2 \rangle / 2)} \left\{ 1 + \pi^{1/2} \vartheta e^{\vartheta^2} [1 + \text{Erf}(\vartheta)] \right\}. \end{aligned} \quad (117)$$

Eq. (116) plays an enhancement factor in the pair-production rate (117).

In the low-temperature limit  $T/|\mu_\phi| \ll 1$ , the distribution (112) shows bosons undergo the Bose-Einstein condensation, by going to the energy level  $\mathcal{E}_\phi = \mu_\phi$ , and the momentum states  $p_\phi^2 = \mu_\phi^2 - m_\phi c^2$ , the leading order of enhancement factor in Eq. (117) is given by

$$\frac{2}{T} (\mathcal{E}_- - \mathcal{E}_+) = \frac{2}{T} (eE_0\ell) > 0. \quad \mathcal{E}_- \geq \mathcal{E}_+ \geq \mu_\phi. \quad (118)$$

We find that the enhancement factor is two, by comparing Eq. (117) with Eq. (76). In the high-temperature limit  $T/|\mu_\phi| \gg 1$  and  $T < (\mathcal{E}_- - \mathcal{E}_+)$  the leading order of Eq. (116) is

$$\frac{2}{T} (\mathcal{E}_- - \mathcal{E}_+) - \ln \frac{\mathcal{E}_- - \mu_\phi}{\mathcal{E}_+ - \mu_\phi} = \frac{2}{T} (eE_0\ell) - \ln \frac{\mathcal{E}_- - \mu_\phi}{\mathcal{E}_+ - \mu_\phi}, \quad (119)$$

and we find that the enhancement factor is

$$2 \left( 1 - \frac{T}{2eE_0\ell} \ln \frac{\mathcal{E}_- - \mu_\phi}{\mathcal{E}_+ - \mu_\phi} \right), \quad (120)$$

by comparing Eq. (117) with Eq. (76).

In the case of Sauter electric field (77) for the semiclassical limit, neglecting  $\mathcal{E}$ -dependence in the prefactor, we consider the following Gaussian  $\mathcal{E}$ -integration (see Fig. 1)

$$\int_{1-\sigma}^{\sigma-1} \frac{d\mathcal{E}}{2\pi\hbar} e^{-\pi(E_c/E_0)(\overline{G}_2 - a_\perp^2 \overline{H}_2) \mathcal{E}^2/2} [1 + f_\phi(\mathcal{E}, \mu_\phi, T)] \approx (\mathcal{E}_- - \mathcal{E}_+) - \frac{T}{2\pi\hbar} \ln \left( \frac{e^{\mathcal{E}_+/T} - e^{\mu_\phi/T}}{e^{\mathcal{E}_-/T} - e^{\mu_\phi/T}} \right) \quad (121)$$

where the exponential factor plays a cutoff given by Eq. (110). To see the Bose-Einstein enhancement, we further neglect  $\mathcal{E}$ -dependence of the exponential factor of Eq. (121), and assume the maximal pair-production probability at  $\mathcal{E} = 0$ , and Eq. (121) approximately becomes

$$\int_{1-\sigma}^{\sigma-1} \frac{d\mathcal{E}}{2\pi\hbar} [1 + f_\phi(\mathcal{E}, \mu_\phi, T)] = 2(\sigma - 1) - \frac{T}{2\pi\hbar} \ln \left( \frac{e^{(\sigma-1)/T} - e^{\mu_\phi/T}}{e^{(1-\sigma)/T} - e^{\mu_\phi/T}} \right). \quad (122)$$

In consequence, the pair-production rate in the Sauter field Eq. (91) multiplied by this enhancement factor (122). We have the same discussions on the high- and low- temperature limits by the replacement  $\sigma - 1 \Rightarrow eE\ell/2$ .

### VIII. THE PRESENCE OF A NEUTRAL PLASMA OF ELECTRONS AND PROTONS

In the previous section, we have considered the pair-production rate in the presence of thermal photons at temperature  $T$  (98). Another physically interesting environment is the presence of a neutral plasma composed of electrons and protons. The two charge components can oscillate against each other and modify the electric field available for pair creation. For simplicity let us assume the protons to form a charged lattice and let us ignore the temperature  $T_{\text{Debye}}$  associated with the lattice phonons. The electrons are distributed in the lattice so as to screen electric fields of the proton charges and of the external electric potential  $A_0$ . In such an equilibrium configuration, we shall assume the electrons to be in a thermal equilibrium at a temperature  $T$  and chemical potential  $\mu_e$ , so that their Fermi distribution is by Eqs. (100) and (101). The associated electron number-density (102), energy-density and pressure

$$\epsilon_e(m_e, \mu_e, T) = 2 \int \frac{d^3 p_e}{(2\pi\hbar)^3} \frac{\mathcal{E}_e}{e^{(\mathcal{E}_e - \mu_e)/T} + 1}, \quad (123)$$

$$P_e(m_e, \mu_e, T) = 2T \int \frac{d^3 p_e}{(2\pi\hbar)^3} \ln \left[ 1 + e^{-(\mathcal{E}_e - \mu_e)/T} \right]. \quad (124)$$

The chemical potential  $\mu_e > 0$  is fixed by the total number  $V n_e(m_e, \mu_e, T)$ . The gradient of electron-gas pressure balances all electric forces.

Due to perturbations, these electrons deviate from their equilibrium positions, and this may lead to the coherent plasma oscillation of electrons in the proton lattice. In order to study this, we first neglect the dissipative terms, and describe perturbation of these electrons as a simple perfect fluid, whose energy-momentum tensor,

$$\delta T_e^{\mu\nu} = \delta P_e g^{\mu\nu} + (\delta P_e + \delta \epsilon_e) U_e^\mu U_e^\nu, \quad (125)$$

where the flat metric  $g^{\mu\nu} = (-, +, +, +)$  and electron four velocity  $U_e^\mu$ . In the energy-momentum tensor (125),  $\delta n_e$ ,  $\delta \epsilon_e$  and  $\delta P_e$  are perturbations of proper number, energy densities and pressure in comoving frame of electron fluid. Such plasma oscillation of electrons around the equilibrium configuration in the proton lattice can be described by the continuity equation, energy-momentum conservation, and the Maxwell equations yield

$$\partial_\nu (\delta n_e U_e^\nu) = 0, \quad (126)$$

$$U_e^\mu \partial_\nu (\delta T_{e\mu}^\nu) = -U_e^\mu \delta F_{\mu\nu} \delta J^\nu, \quad (127)$$

$$\partial_\nu (\delta F^{\mu\nu}) = -4\pi \delta J^\mu, \quad (128)$$

where  $\delta F_{\mu\nu}$  is the strength of fluctuation electromagnetic fields due to the fluctuating electric four current

$$\delta J^\mu = e(\delta n_p U_p^\nu - \delta n_e U_e^\nu). \quad (129)$$

Here  $n_p$  is the proton number-density, and  $U_p^\nu = (1, 0, 0, 0)$  four-velocity of the protons. On the r.h.s. of Eq. (127), the dissipative term

$$e\delta n_p U_e^\mu U_p^\nu \delta F_{\mu\nu}, \quad (130)$$

indicates an Ohmic heating  $\delta Q$ , and we assume that this term is negligible for  $\delta n_p \approx 0$ ,  $\delta Q = \delta S/T \approx 0$  and the entropy  $S$  is approximately conserved ( $\delta S \approx 0$ ). This is consistent with non dissipative energy-momentum tensor (125) we have adopted for electrons. In consequence, the energy-momentum conservation along four-velocity  $U_e^\mu$ , i.e.,  $U_e^\mu \partial_\nu (\delta T_{e\mu}^\nu) = 0$ , gives the first law of thermodynamics in the form

$$\delta P_e + \delta \epsilon_e = \mu_e \delta n_e, \quad (131)$$

corresponding to the equation of state  $\delta P_e / \delta \epsilon_e = \kappa^2 = \text{const}$ , for a isothermal process of constant temperature  $T$ .

As discussed, the electrons deviate from their equilibrium positions, thereby creating a small electric potential  $\delta A_0$ , associated with the fluctuating electromagnetic field  $\delta F^{\mu\nu}$  in Eq. (128). The perturbed electron distribution  $f_e(\mathcal{E}_e, \mu_e, T, \delta A_0)$  is given by the replacement

$$\mathcal{E}_e \rightarrow \mathcal{E}_e - e\delta A_0, \quad (132)$$

in the electron distribution (100). Expanding perturbed electron distribution  $f_e(\mathcal{E}_e, \mu_e, T, \delta A_0)$  up to the leading order  $\delta A_0$ , we obtain

$$f_e(\mathcal{E}_e, \mu_e, T, \delta A_0) \approx f_e(\mathcal{E}_e, \mu_e, T) \left[ 1 + \frac{e}{T} \delta A_0 e^{(\mathcal{E}_e - \mu_e)/T} f_e(\mathcal{E}_e, \mu_e, T) \right], \quad (133)$$

and an electron number-density fluctuation

$$\begin{aligned} \delta n_e(m_e, \mu_e, T) &\approx \frac{2e}{T} \delta A_0 \int \frac{d^3 p_e}{(2\pi\hbar)^3} \frac{e^{(\mathcal{E}_e - \mu_e)/T}}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} \\ &= \frac{e}{T} \delta A_0 \left[ n_e(m_e, \mu_e, T) - 2 \int \frac{d^3 p_e}{(2\pi)^3} \frac{1}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} \right], \end{aligned} \quad (134)$$

as well as energy-density fluctuation

$$\begin{aligned} \delta \epsilon_e(m_e, \mu_e, T) &\approx \frac{2e}{T} \delta A_0 \int \frac{d^3 p_e}{(2\pi\hbar)^3} \frac{\mathcal{E}_e e^{(\mathcal{E}_e - \mu_e)/T}}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} - e\delta A_0 n_e(m_e, \mu_e, T) \\ &= \frac{e}{T} \delta A_0 \left[ \epsilon_e(m_e, \mu_e, T) - T n_e(m_e, \mu_e, T) - 2 \int \frac{d^3 p_e}{(2\pi)^3} \frac{\mathcal{E}_e}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} \right]. \end{aligned} \quad (135)$$

This yields an electron pressure fluctuation

$$\begin{aligned}\delta P_e(m_e, \mu_e, T) &\approx 2e\delta A_0 \int \frac{d^3 p_e}{(2\pi\hbar)^3} \frac{e^{-(\mathcal{E}_e - \mu_e)/T}}{[e^{-(\mathcal{E}_e - \mu_e)/T} + 1]} \\ &= e\delta A_0 n_e(m_e, \mu_e, T),\end{aligned}\quad (136)$$

which propagates through the electron gas.

In order to study the propagation of such plasm, we consider the Maxwell equation (128) for the fluctuation field  $\delta A_0$  caused by the charge fluctuations

$$\nabla^2 \delta A_0 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \delta A_0 = -4\pi e[\delta n_p - \delta n_e], \quad (137)$$

where the velocity is given in units of the speed of light in vacuum  $c$ :

$$v^2 = \frac{\delta P_e}{\delta \epsilon_e} = \frac{T n_e(m_e, \mu_e, T)}{\left[ \epsilon_e(m_e, \mu_e, T) - T n_e(m_e, \mu_e, T) - 2 \int \frac{d^3 p_e}{(2\pi)^3} \frac{\mathcal{E}_e}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} \right]}. \quad (138)$$

This is a constant  $\kappa^2$  in an isothermal process of constant temperature  $T$ . We shall ignore the much smaller fluctuations of the proton distribution  $\delta n_p \approx 0$ . Inserting Eq. (134) and a plane wave ansatz  $\delta A_0 = e^{-i\omega t + i\mathbf{k}\mathbf{x}}$ , we obtain an the energy-spectrum for the plasma waves

$$\omega_k^2 \equiv \omega_{\text{pl}}^2(|\mathbf{k}|) = \omega_{\text{pl}}^2 + v^2 |\mathbf{k}|^2, \quad (139)$$

where

$$\begin{aligned}\omega_{\text{pl}}^2 &\equiv \frac{\alpha^2 c^3}{T} \int \frac{d^3 p_e}{\pi^2} \frac{e^{(\mathcal{E}_e - \mu_e)/T}}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2}, \\ &= \frac{2\pi\alpha^2 c^3}{T} \left[ n_e(m_e, \mu_e, T) - 2 \int \frac{d^3 p_e}{(2\pi)^3} \frac{1}{[e^{(\mathcal{E}_e - \mu_e)/T} + 1]^2} \right]\end{aligned}\quad (140)$$

is the *plasma frequency* of the electron gas in the proton lattice. These plasma oscillations propagate through the plasma with a transverse electromagnetic wave  $\mathbf{A}_\perp(x)$  with two transverse polarizations. Their propagator is given by

$$\frac{\delta_{ij} - k_i k_j / |\mathbf{k}|^2}{\omega_k^2 - \omega_{\text{pl}}^2(\mathbf{k})}, \quad (141)$$

that we call plasmaon field whose energy dispersion is given by (139), corresponding to massive photons. Their excitation energies will be in thermal equipartition with the thermal state of electrons in the same temperature  $T$ . In consequence, the thermal distribution function of these massive photons is given by Eq. (98) with the energy dispersion-relation  $\omega_{\text{pl}}^2(|\mathbf{k}|)$  of (139). Following



the same calculations from Eqs. (97-99), we calculate the average of  $a_{\perp}^2 = (e\mathbf{A}_{\perp})^2/(m_e^2 c^4)$  [Eq. (50)] of massive photon fields  $\mathbf{A}_{\perp}$  in thermal plasma state,

$$\begin{aligned} \frac{1}{2}\langle a_{\perp}^2 \rangle &= \frac{\alpha}{2m_e^2 c^4} \int \frac{d^3 k}{(2\pi)^3 \omega_{\text{pl}}^2(|\mathbf{k}|)} f_{\text{pl}}(k) \\ &= \frac{\alpha}{2m_e^2 c^4} \int \frac{d^3 k}{(2\pi)^3 \omega_{\text{pl}}^2(|\mathbf{k}|)} \frac{2}{e^{\omega_{\text{pl}}^2(|\mathbf{k}|)/T} - 1}. \end{aligned} \quad (142)$$

For the case that temperature  $T$  is much larger than the plasma frequency  $\omega_{\text{pl}}$ , Eq. (142) is approximately equal to Eq. (99), while for the case that  $T$  is much smaller than the plasma frequency  $\omega_{\text{pl}}$ , Eq. (142) is approximately proportional to  $\alpha(\omega_{\text{pl}}\hbar/m_e c^2)^2$ .

It is interesting to discuss the case that a monochromatic electromagnetic wave (6), (93)-(95) propagates through the plasma of electrons in the proton lattice. Define a dielectric constant  $\epsilon = 1 + \chi_e$ , where the susceptibility  $\chi_e$  is given by Eqs. (137) and (139)

$$\chi_e = -\frac{\omega_{\text{pl}}^2}{\omega^2 - v^2|\mathbf{k}|^2}, \quad (143)$$

as a function of the frequency  $\omega$  and wave-vector  $|\mathbf{k}|$  of the monochromatic electromagnetic wave (laser beam) propagating in the plasma. The displacement field strength in the plasma  $\mathbf{D} = \epsilon\mathbf{E}$ . For large frequencies  $\omega^2/|\mathbf{k}|^2 \gg v^2$ ,  $\chi_e \approx -\omega_{\text{pl}}^2/\omega^2$ , the dielectric constant  $\epsilon \approx 1 - \omega_{\text{pl}}^2/\omega^2$  and  $\epsilon \approx 1$  for  $\omega^2 \gg \omega_{\text{pl}}^2$ . While for small frequencies  $\omega^2/|\mathbf{k}|^2 \ll v^2$ ,  $\chi_e \approx +\omega_{\text{pl}}^2/|\mathbf{k}|^2 v^2$ , the dielectric constant  $\epsilon \approx 1 + \omega_{\text{pl}}^2/(|\mathbf{k}|^2 v^2)$ . The resonance appears at  $\omega^2 = \omega_{\text{res}}^2 \equiv |\mathbf{k}|^2 v^2$ , at which the dielectric constant  $|\epsilon| \gg 1$ , and the displacement field  $\mathbf{D}$  greatly increases.

There are no imaginary damping terms in the denominators of Eqs. (141) and (143), because we use the perfect fluid stress tensor (125) for the electron plasma. In particular we neglect Ohmic heating in Eq. (127). If take this into account in (130), we have the following energy dissipation per electron in a period  $\mathcal{T} = 2\pi/\omega$ :

$$\delta\mathcal{E}_{\text{diss}} = -e\mathcal{T} \left( \frac{\delta n_p}{n_e} \right) \left( \frac{\delta x_e^i}{\delta\tau} \right) \left( \frac{\delta A_0}{\delta x_e^i} \right) = -e\mathcal{T} \left( \frac{\delta n_p}{n_e} \right) \gamma \left( \frac{\delta A_0}{\delta t} \right) = i(2\pi)e \left( \frac{\delta n_p}{n_e} \right) \gamma \delta A_0, \quad (144)$$

where  $\gamma \approx 1$  is a Lorentz factor. This small dissipative term  $\delta\mathcal{E}_{\text{diss}}$  should be added into Eq. (132), namely, replacing the energy perturbation  $e\delta A_0$  by

$$e\delta A_0 + \delta\mathcal{E}_{\text{diss}} = e\delta A_0 \left[ 1 + i(2\pi) \left( \frac{\delta n_p}{n_e} \right) \gamma \right]. \quad (145)$$

This creates an imaginary damping term in the denominators of Eqs. (141) and (143), limiting the life time of plasmons via a finite width of the resonance.

However, a great increase of displacement field  $\mathbf{D}$  at the resonance for  $\omega^2 = \omega_{\text{res}}^2$  does not yet enhance the pair-production rate. The expectation  $\frac{1}{2}\langle a_{\perp}^2 \rangle$  in Eq. (95) for doing this is purely due

to electric field  $\mathbf{E}$  of laser beams, and ultrahigh-intensity laser beams are required. Help can come from the self-focusing phenomenon of ultrahigh-intensity laser beams propagating in the plasma of electrons and protons. These can be used in principle realize also a large electric field, and thus a large term  $\frac{1}{2}\langle a_{\perp}^2 \rangle$  (95). If laser intensities are larger than a certain threshold critical power (see review [24])

$$P_{\text{cr}} = \frac{m_e c^5 \omega^2}{e^2 \omega_{\text{pl}}^2} \simeq 17 \left( \frac{\omega}{\omega_{\text{pl}}} \right)^2 \text{ GW}, \quad (146)$$

for relativistic self-focusing, the laser pulse can be self-focused when propagating through a plasma of electrons and protons with the plasma frequency  $\omega_{\text{pl}}$  (140). It will be interesting to measure the electron-positron pair production by a self-focused ultrahigh-intensity laser beam in such an environment.

To end this section we note that if electrons were bosons, one can do calculations by using the Bose-Einstein distribution instead of Fermi one. The discussions and conclusions are the same. The total pair-production rate receives a factor of suppression and enhancement that are discussed in the previous Secs. VI and VII.

## IX. SUMMARY AND REMARKS

In Ref. [12], we studied the process of electron-positron pair production from the vacuum as a quantum tunneling phenomenon, we derived in semiclassical approximation the general rate formulas (55) and (56) with  $\frac{1}{2}a_{\perp}^2 = 0$ . They consist of a Boltzmann-like tunneling exponential, and a pre-exponential factor, and are applicable to any system where the field strength points mainly in one direction and varies only along this direction. In this article, we generalize these formulas to the presence of a monochromatic electromagnetic wave (6), (93)-(95) in addition to a static electric field in one direction. We have also considered the system of electrons and charged bosons at finite temperature and chemical potential. In several cases, we calculate and discuss the factor  $\langle \frac{1}{2}a_{\perp}^2 \rangle$  for enhancing pair-production rate. In particular, we consider the plasma of electrons and protons, and point out the self-focusing phenomenon of ultrahigh intensity laser beams in the plasma possibly gives rise to a larger factor  $\langle \frac{1}{2}a_{\perp}^2 \rangle$  for enhancing pair-production rate, and this could be experimentally relevant for observing pair production in laboratories.

Apart from its purely theoretic interest, these formulas for the pair-production rate in one-direction nonuniform fields and laser fields can be relevant for studying the collisions of laser beams [13] and heavy ions [25–27], an explanation of the powerful Gamma Ray Bursts in astrophysics

[5, 28], and strong electric fields in the surface shell of compact stars [29]. It is also interesting to use these formulas to study the phenomenon of plasma oscillations [30] of electrons and positrons after their creation in electric fields.

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